Generalized Complex Geometry and the Poisson Sigma Model

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Non-linear Sigma Model and Complex Geometry

Gates, Hull, Rocek, NPB238:349 (1984)

$$\mathcal{S} = \frac{1}{2}\int d^2x d^2\theta (g_{ij}(X) + B_{ij}(X)) D_+ X^i D_- X^j$$

has manifest $N = (1,1)$ SUSY.

Additional non-manifest $N = (2,2)$ SUSY

$$\delta_+ X^I = \varepsilon^+ D_+ X^j I^{+i}{}_j$$
$$\delta_- X^I = \varepsilon^- D_- X^j I^{-i}{}_j$$

with the conditions

1. g hermitian with respect to $I^\pm$.
2. $I^\pm$ are complex structures.
3. $I^\pm$ are covariantly constant.
4. $[I^+, I^-] \neq 0$ in general.

Generalized Complex Geometry

Hitchin, math.DG/0209099; Gualtieri, math.DG/0401221

A generalized complex structure is a mapping $\mathcal{J}$ from $T \oplus T^* \to T \oplus T^*$ such that

- $\mathcal{J}^2 = -1$,
- The pseudo-metric $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is hermitian with respect to $\mathcal{J}$,
- $\mathcal{J}$ obeys an integrabitlity condition with resepct to the Courant bracket on $T \oplus T^*$:
$$[\mathcal{X}, \mathcal{Y}]_C - [\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}]_C + \mathcal{J}[\mathcal{J}\mathcal{X}, \mathcal{Y}]_C + \mathcal{J}[\mathcal{X}, \mathcal{J}\mathcal{Y}]_C = 0$$
- The Courant bracket can be twisted by a closed three form H.

Decomposition

$$\mathcal{J} = \begin{pmatrix} J : T \to T & P : T^* \to T \\ L : T \to T^* & K : T^* \to T^* \end{pmatrix}$$

Limiting cases:

1. $L = P = 0 \longrightarrow J = -K^T$ complex structure.
2. $J = 0 \to K = 0 \longrightarrow P = -L^{-1}$ symplectic structure.

Relation to Sigma Model

Gualtieri, math.DG/0401221

The data encoded in the choice

$$(g, H = \mathrm{d}B, I^+, I^-)$$

equivalent to two H-twisted generalized complex structures on M with

$$\mathcal{J}_{1/2} = \frac{1}{2}\begin{pmatrix} I^+ \pm I^- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I^+ \pm I^-)^T \end{pmatrix}$$

$\omega_\pm = g I^\pm$.

It follows

$$[\mathcal{J}_1, \mathcal{J}_2] = 0$$

and

$$G = -q\mathcal{J}_1\mathcal{J}_2$$

defines a positive definite metric on $T \oplus T^*$.

Traget space is twisted generalized Kähler.

First Order Action

Lindström, hep-th/0401100;

Lindström, Minasian, Tomasiello, Zabzine, hep-th/0405085;

LB, hep-th/0409283

$$\mathcal{L}_{FO} = \frac{1}{2}\int \mathrm{d}^2\theta \Big(i(D_- X^i) A_{i+} - i(D_+ X^i) A_{i-} + (\mathcal{G} + \mathcal{P})^{ij} A_{j+} A_{i-} \Big)$$

Additional non-manifest SUSY

$$\delta_+ X^i = \varepsilon^+ D_+ X^j J^{+i}{}_j + \varepsilon^+ A_{j+} P^{+ij}$$

$$\delta_+ A_{i+} = \sqrt{2}\varepsilon^+ \partial_{++} X^j L^+_{ij} - \varepsilon^+ D_+ A_{j+} K_i^{+j} + \dots$$

$$\delta_+ A_{i-} = \varepsilon^+ D_+ A_{j-} R_i^{+j} + \varepsilon^+ D_+ D_- X^j T^+_{ij} + X^j A_{k-} V_{ij}^{+k} \dots$$

Problem: If one requires on-shell closure only
$\rightarrow$ ensuing relations depend solely on R.

Off-shell Closure

Check $\delta^1_+\delta^2_+ X^i = \sqrt{2}\varepsilon^+_1\varepsilon^+_2\partial_{++}X^i$.

- J, P, L, K can be chosen to form a GCS.
- All coefficients in $\delta_+ X^i$ and $\delta_+ A_{i+}$ expressed through GCS.
- $\delta^1_+\delta^2_+ A_{i+} = \sqrt{2}\varepsilon^+_1\varepsilon^+_2\partial_{++}A_{i+}$ fulfilled automatically.
- $\delta_+ A_{i-}$: R is almost complex. T, V are the remaining undetermined coefficients.
- Conditions relating GCS to $\mathcal{G}$ and $\mathcal{P}$:

$$P^{ij} = -i(\mathcal{P}K)^{[ij]} - (\mathcal{P}L\mathcal{P})^{ij} - (\mathcal{G}L\mathcal{G})^{ij}$$

$$i(\mathcal{G}K)^{\{ij\}} + (\mathcal{P}L\mathcal{G})^{ij} + (\mathcal{G}L\mathcal{P})^{ij} = 0$$

and differential conditions.

$\{Q_+, Q_+\}A_{i-} =??$ $\{Q_+, Q_-\}\Psi =??$

Poisson Sigma Model

LB, hep-th/0409283

Special case: $\mathcal{G} = 0$, $\mathcal{P}$ Poisson.
Casimir-Darboux coordinates: $\mathcal{P}$ constant anti-symmetric.

- Poisson structures:
$$P^{\pm} = -i\mathcal{P}K^{\pm} - iJ^{\pm}\mathcal{P} - \mathcal{P}L^{\pm}\mathcal{P}$$

- off-shell closure: Complicated differential relations, no obvious geometric interpretation.

- on-shell closure: Remaining conditions depend on two almost complex structures $R^{\pm}$.

- $\mathcal{P}$ symplectic: $\mathcal{P}R^{\pm} = I^{\pm}\mathcal{P}$, $I^{\pm}$ complex structure

- $\mathcal{P}$ Poisson: In certain cases $R^{\pm}$ related to complex structure on symplectic leaf.

- $[R^{+}, R^{-}] = 0$, but $[\mathcal{J}^{+}, \mathcal{J}^{-}] \neq 0$.

Conclusions

- On-shell closure: first-order action does not yield GCS.

- Off-shell closure:

  – First-order action in general needs more structure than GCS.

  – Other realizations of GCS? Lindström, Rocek, Unge, Zabzine, hep-th/0411186.